Induced gap solitons of a Korteweg–de Vries system

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We consider a KdV system where a periodic cnoidal wave is initially excited. It is found that envelope solitons can be formed by perturbation modes, which build up on the periodic wave and move relative to it. The mechanism is similar to that of the so-called gap soliton, where the existing cnoidal wave plays the role of a periodic structure. [S1063-651X(98)10912-1]

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I. INTRODUCTION

In recent years nonlinear wave propagation through inhomogeneous media has attracted considerable attention [1,2]. One of the simplest and physically relevant models of an inhomogeneous continuous medium is that with periodic variations on its parameters. In these systems, a new type of nonlinear excitations, so-called gap solitons, discovered in 1987 by Chen and Mill [1], can be built up due to the cooperation between two factors: periodicity of the parameters and nonlinearity of the system. Periodicity of the system parameter is induced, for example, by a periodic change of the linear refractive index in optical waveguide or fibers with a Kerr-type nonlinearity. Gap solitons may also exist as localized structures in nonlinear discrete periodic media, such as a diatomic chain [3]. The common feature of gap solitons in the models mentioned above is: a wave lying in a gap of the linear spectrum will be reflected strongly, but spatially localized modes may appear when the nonlinearity is considered. The more physical explanation for the existence of gap solitons is: periodic variation of the parameter supplies a feedback mechanism [4], and then there are two counterpropagating waves in the system. When the nonlinearity is considered, these waves are coupled and nonlinear excitation can emerge in certain cases. Therefore, the difference between gap solitons and usual solitons lies in the fact that the appearance of the former requires periodic variation of the parameter.

In a coupled Korteweg–de Vries (KdV) wave system [6], there is no prerequisite periodic structure, but the gap solitons can also be found. In fact, the necessary periodic potential for one subsystem is provided by the other because of a weak linear coupling between the two subsystems, which opens a narrow gap in the linear spectrum. A similar phenomenon has also been observed in a system with only one equation. For example, in Ref. [9] with a driven/damped nonlinear drift wave equation, the gap solitary wave is numerically found to be trapped by a steady wave solution. In this case, the steady wave solution plays the role of a periodic background. In the present work, we analytically discuss such phenomena occurring in a conservative system, here KdV equation as an example.

As is well known, the KdV equation allows for a periodic cnoidal wave solution. Supposing that initially such a cnoidal wave already exists, it is then interesting to know what kind of effect it will have on the system. If there is no prerequisite periodic wave, the linear dispersion of the normal KdV equation tells us that the system allows for a unique propagation direction for harmonic perturbation waves. However, the situation will be changed when there already exists a periodic solution, e.g., a cnoidal wave. In this case, one can examine the dispersive relation and find that the system allows for two counterpropagating waves moving relative to the existing cnoidal wave. The reason is that the perturbation waves can be scattered by the periodic wave as if the latter is a periodic potential. In the present paper we will show that the nonlinearity can cause the two counterpropagating waves to be coupled and an envelope soliton can be formed. We call it an induced gap soliton in the sense that it is fully supported by the background cnoidal wave and it behaves just like the gap solitons appearing in a periodic medium.

We consider the case in which the amplitude of the background cnoidal wave is small. The dispersion of our system is discussed in Sec. II. In Sec. III, by using the asymptotic expansion technique (see Ref. [5]), the induced gap solitons are found. Finally, Sec. IV includes the conclusions and discussions.

II. DISPERSION

We consider the KdV equation

$$\varphi_t-\varphi_{xxx}-\varphi\varphi_x=0, \tag{1}$$

where the subscripts x,t denote the derivations with respect to space and time variables, respectively. It is well known that this equation has a periodic cnoidal wave solution

$$\begin{aligned}\varphi_0(x-vt)&=A\,\mathrm{sn}^2(x-vt|p)\\&=A_0+\sum_{n=1}^{\infty}A_n\cos[2nG(x-vt)]\equiv A_0+\tilde{\varphi},\end{aligned} \tag{2}$$

where p is the modulus, $A=-12q$, $v=4+4q$, $q\equiv p^2$, $A_0=-12(1-E/K)$, $A_n=12\pi^2 n/[K^2\sinh(n\pi K'/K)]$, K, K', E are the complete elliptic integrals, and $G=\pi/2K(p)$. This solution is periodic and its period is $2K(p)$, when $p\to0$, $K(p)\to\pi/2$, $K'\to\infty$; when $p\to1$, $K(p)\to\infty$, $K'\to\pi/2$. Assuming that such a periodic solution φ_0 has been excited in our system, we set $\varphi(x,t)=\varphi_0(x-vt)+\psi(x,t)$ and then get an equation for the perturbation wave ψ,

$$\psi_t-\psi_{xxx}-(\varphi_0\psi)_x-\psi\psi_x=0.$$

If $\varphi_0\to 0$, the above equation is the KdV system, and its linear dispersive relation is $\omega=-k^3$ or $\omega/k=-k^2<0$. It tells us that the KdV system allows for a unique propagation direction, here the negative x direction. A finite excitation $\varphi_0(x)$, however, dramatically changes the dispersion of this system. Since $\varphi_0(x-vt)$ moves with a velocity v, we transform the reference frame to $x'=x-vt$, $t'=t$. Then for the perturbation waves moving relative to φ_0, the latter is felt as a periodic potential and will scatter those perturbation waves. As a result, two counter propagation directions are allowed for the harmonic waves of ψ. It is those counterpropagating perturbation waves moving relative to the periodic potential φ_0 that can form localized structures. These structures are different from the usual KdV solitons, and can be called induced gap solitons as explained in the following.

Omitting the prime "′" in the reference frame (x', t'), one obtains

$$\psi_t-\psi_{xxx}-\tilde{v}\psi_x-(\tilde{\varphi}\psi)_x-\psi\psi_x=0, \tag{3}$$

where $\tilde{v}=v+A_0$. First let us look for the nonlinear dispersion caused by the periodic potential $\varphi_0(x)$. For this purpose, one sets the nonlinear term to zero, and expands $\psi(x,t)$ into $\psi(x,t)=\Sigma_k\int d\omega u(k,\omega)e^{i(\omega t-kx)}$. Inserting it into Eq. (3) brings us the following equation:

$$u(k,\omega)(\omega+k^3-vk)-\sum_{n=0}^{\infty}\tfrac{1}{2}fA_nk \times[u(k+2nG,\omega)+u(k-2nG,\omega)]=0, \tag{4}$$

where $k=1, 2, \ldots, \infty$. The condition that the Jacobian determinant of the above equations equals zero gives the dispersive relation of $\psi(x,t)$. Obviously, the system dispersion is significantly influenced by the cnoidal wave $\varphi_0(x)$. The most important effect is that $u(k,\omega)$ is allowed to propagate in both directions.

We consider the case in which q is small, i.e., $q\sim\epsilon$, ϵ being a small parameter. When $q=p^2$ is small, $E\to\pi/2$, $K\to\pi/2$, K' is very large. So $G\sim\epsilon^0$, $|A_0|\sim\epsilon$, $A_1\to\exp[-(\pi K'/K)]\sim\epsilon$, and $A_2\to\exp[-(2\pi K'/K)]\sim\epsilon^2$. For simplicity, we only consider the case where the wave numbers of the harmonics of ψ equal halves of those of periodic potential φ_0, i.e., nG. So we expand the wave field ψ into two counter propagating modes. That is

$$\psi(x,t)=\sum_n \epsilon^n[a_n^{(+)}(x,t)\mathrm{e}^{inG(x-ct)}+a_n^{(-)}(x,t)\mathrm{e}^{-inG(x+ct)}] +\text{c.c.}, \tag{5}$$

where $a_n^{(+)}$ represents the amplitude of the forward-moving mode along the positive x direction, and $a_n^{(-)}$ is the backward-moving modes when $c>0$. c.c. stands for complex conjugation and ϵ is the small parameter. We assume the slowly varying envelope approximation; i.e., the amplitudes a_n are considered to be slowly varying in x and t or $\partial/\partial x\sim\epsilon$, $\partial/\partial t\sim\epsilon$ when operating on a_n. Substituting the expansion (5) into Eq. (3), we obtain the following equations for $a_1^{(+)}$ and $a_1^{(-)}$:

$$\frac{\partial a_1^{(+)}}{\partial t}-\delta\frac{\partial a_1^{(+)}}{\partial x}-i(d_1+Gc)a_1^{(+)}-\frac{A_1}{2}a_1^{(-)}(iG) -\tfrac{1}{2}a_2^{(+)}[a_1^{(+)}]^*(iG)=0, \tag{6a}$$

$$\frac{\partial a_1^{(-)}}{\partial t}-\delta\frac{\partial a_1^{(-)}}{\partial x}+i(d_1-Gc)a_1^{(-)}+\frac{A_1}{2}a_1^{(+)}(iG) +\tfrac{1}{2}a_2^{(-)}[a_1^{(-)}]^*(iG)=0, \tag{6b}$$

where $\delta=-3G^2+\tilde{v}$, and $d_1=-G^3+G\tilde{v}$. It is important to note that in Eqs. (6a) and (6b) nonlinear parts also involve the amplitudes of the second harmonic, which after substitution are reduced in the lower order of ϵ to the simple algebraic relations

$$a_2^{(+)}=-\frac{[a_1^{(+)}]^2}{2(d_2+c)},\quad a_2^{(-)}=-\frac{[a_1^{(-)}]^2}{2(d_2-c)}, \tag{7}$$

where $d_2=-4G^2+\tilde{v}$. Using Eqs. (7), Eqs. (6a) and (6b) change to

$$\frac{\partial a_1^{(+)}}{\partial t}-\delta\frac{\partial a_1^{(+)}}{\partial x}-i(d_1+Gc)a_1^{(+)}-\frac{A_1}{2}a_1^{(-)}(iG) +\frac{1}{4(d_2+c)}|a_1^{(+)}|^2[a_1^{(+)}](iG)=0, \tag{8a}$$

$$\frac{\partial a_1^{(-)}}{\partial t}-\delta\frac{\partial a_1^{(-)}}{\partial x}+i(d_1-Gc)a_1^{(-)}+\frac{A_1}{2}a_1^{(+)}(iG) -\frac{1}{4(d_2-c)}|a_1^{(-)}|^2[a_1^{(-)}](iG)=0. \tag{8b}$$

If the last nonlinear dispersive terms are neglected, nonzero solutions of $a_1^{(+)}$ and $a_1^{(-)}$ require

$$c=[(d_1/G)^2-A_1^2/4]^{1/2}. \tag{9}$$

This gives the dispersive relation when initially a cnoidal wave exists. This dispersion can also be obtained from Eq. (4) by omitting the second and higher Fourier component of $\varphi_0(x)$. When $A_1\to 0$, the linear dispersion of KdV equation is recovered. If A_1 is nonzero, those harmonic waves whose velocities c do not satisfy the relation (9) are forbidden by the system. This dispersion produces a gap in the spectrum. However, one can see in the next section that the nonlinearity causes the modes to be coupled to form localized structures. From this point of view, these localized structures can also be called gap solitons.

III. INDUCED GAP SOLITONS

To look for solutions of nonlinear equations (8a) and (8b), we consider the envelopes $a_1^{(+)}$, $a_1^{(-)}$ propagating with a velocity w:

$$a_1^{(+)}(x,t)=U(x-wt), \quad a_1^{(-)}(x,t)=V(x-wt). \tag{10}$$

After transforming to the moving coordinates $x'=x-wt$, $t'=t$, and inserting Eq. (10) into Eqs. (8a) and (8b), we obtain the following set of ordinary differential equations:

$$-(\delta+w)U_{x'}-i(d_1+Gc)U+\frac{iG}{4(d_2+c)}|U|^2U=i\frac{A_1}{2}GV, \tag{11a}$$

$$-(\delta+w)V_{x'}+i(d_1-Gc)V-\frac{iG}{4(d_2-c)}|V|^2V=-i\frac{A_1}{2}GU. \tag{11b}$$

U,V are both complex functions. In the case of $c=0$, Eq. (11b) is conjugate to Eq. (11a). Now we assume $U=u+iv$, $V=u-iv$; then the equations for u, v are the same as those describing the optical bright and dark gap solitons in a diffractive $\chi^{(2)}$ medium [5] with periodic variation of refractive index. So the details are not listed here. However, the physics is different. In our case, the gap solitons are induced, supported by the normal cnoidal wave of the KdV equation.

In the general case for $c\neq 0$, we set

$$U(x')=R_1(x')\exp[i\alpha(x')], \quad V(x')=R_2(x')\exp[i\beta(x')], \tag{12}$$

where R_1, R_2, α, and β are real functions. Exponential factors in Eq. (12) and $\exp[inG(x-ct)]$ in Eq. (5) constitute the high-frequency part of $\psi(x)$. $R_1(x')$, $R_2(x')$ represent the envelopes of $\psi(x)$. Substituting Eq. (12) into Eqs. (11a) and (11b), one gets four coupled equations for R_1, R_2, α, and β as follows:

$$(\delta+w)R_{1x'}=\tfrac{1}{2}A_1GR_2\sin(\beta-\alpha), \tag{13a}$$

$$(\delta+w)R_{2x'}=\tfrac{1}{2}A_1GR_1\sin(\beta-\alpha), \tag{13b}$$

$$-(\delta+w)R_1\alpha_{x'}-(d_1+Gc)R_1+\frac{G}{4(d_2+c)}R_1^3=\tfrac{1}{2}A_1GR_2\cos(\beta-\alpha), \tag{13c}$$

$$-(\delta+w)R_2\beta_{x'}+(d_1-Gc)R_2-\frac{G}{4(d_2-c)}R_2^3=-\tfrac{1}{2}A_1GR_1\cos(\beta-\alpha). \tag{13d}$$

From Eqs. (13a) and (13b), one obtains a first integral $R_1^2=R_2^2+\text{const}$. We consider the simplest case $R_1=R_2=R$, i.e., we choose const$=0$, and set $\Phi=\beta-\alpha$. After subtracting Eq. (13c) from Eq. (13d), two equations for R and Φ are

$$-(\delta+w)\Phi_{x'}+2d_1-\frac{d_2G}{2(d_2^2-c^2)}R^2=-A_1G\cos\Phi,$$

$$(\delta+w)R_{x'}=\frac{1}{2}A_1GR\sin\Phi.$$

Multiplying R on both sides of the second equation and denoting $S\equiv R^2=|U|^2=|V|^2$, we rewrite the above equations as

$$-(\delta+w)\Phi_{x'}+2d_1-\frac{d_2G}{2(d_2^2-c^2)}S=-A_1G\cos\Phi, \tag{14a}$$

$$(\delta+w)S_{x'}=A_1GS\sin\Phi. \tag{14b}$$

A first integral of above equations can be found:

$$E=hS-eS^2+fS\cos\Phi, \tag{15}$$

in which E is an integration constant and $h=2d_1$, $e=d_2G/[2(d_2^2-c^2)]$, $f=A_1G$. When q is small, $h>f>0$. So from Eq. (15) and Eqs. (14a) and (14b), we get the following equation for S only:

$$(\delta+w)\frac{dS}{dx'}=e[(S-a4)(a1-S)(a2-S)(a3-S)]^{1/2}, \tag{16}$$

where $a1\times a4=a2\times a3=E/|e|>0$, $a1+a4=\frac{1}{2}(f+h)/|e|$, $a2+a3=\frac{1}{2}(h-f)/|e|$, and $a1>a2>a3\geqslant S>a4>0$. Therefore, we obtain the solution of Eq. (16) in terms of the Jacobian elliptic function,

$$R(x-wt)=[S(x-wt)]^{1/2}=\left[a1-\frac{a1-a4}{\frac{a1-a4}{a1-a3}\mathrm{sn}^2[c_s(x-wt),r]+1}\right]^{1/2}, \tag{17}$$

where

$$c_s=\frac{|e|\sqrt{(a1-a3)(a2-a4)}}{2(\delta+w)}, \quad r=\sqrt{\frac{(a1-a2)(a3-a4)}{(a1-a3)(a2-a4)}}.$$

We do not display the expression for net phase $\beta-\alpha$, as it is not needed here. When the background wave $\varphi_0(x)$ is given, the solitary wave described by Eq. (17) is determined by the modulus r. When $r\to 1$, i.e., $a2=a3$, $\mathrm{sn}^2\to\mathrm{sech}^2$, it has a shape of soliton. As the velocity w gets larger, c_s becomes smaller and then the soliton's width gets larger. Its amplitude, however, is independent of w. These properties, due to the background wave, are different from the usual KdV soliton.

IV. CONCLUSION AND DISCUSSION

We discuss the gap solitary wave in a KdV system where initially a periodic cnoidal wave solution exists. Although the amplitude of the periodic potential is small, it has an influence on the perturbation waves propagating in the system. First, the dispersion relation has been changed that allows for two counterpropagating waves moving relative to the background cnoidal wave. Then the two counter propagating waves can be coupled due to the nonlinearity. As a result, gap solitary waves can be formed. Its envelope is a function of space and time that has a shape of a cnoidal wave described by Eq. (17). Our solution is similar to that found in the discretized KdV equation [8] by applying the formalism of dispersion equations.

Moreover, an induced gap soliton, combined with the background cnoidal wave $\varphi_0(x)$, forms a new solution of the KdV system. We notice that in Ref. [7], A. R. Osborne discussed the numerical construction of nonlinear wave train solutions to the periodic KdV equation. These solutions were represented by a linear superposition of nonlinear interacting hyperelliptic functions that are the nonlinear oscillation modes of the equation, and the amplitude of the nonlinear modes are constants of motion for KdV evolution. They constitute the basis functions of the expansion. Integrability of the equation is a necessary condition for the approach. Our solution of the KdV system represented here in its form is also a linear superposition of two nonlinear waves, i.e., the normal cnoidal wave and the induced gap solitary wave. But, in contrast to Ref. [7], in our derivation the two waves are not required as constants of motion. So exact integrability of the equation is not a necessary condition, and we may apply the method in other nonintegrable systems as well, to discover the induced gap solitons.

Here we should stress that the gap solitary wave in our system is supported by and coexists with the usual cnoidal wave. This is different from the gap soliton found in the coupled KdV wave system [6] where the usual KdV solitons are killed by coupling. We would expect the mechanism revealed in the present work can be very general and it may play an important role in the complex dynamics of nonlinear systems. For instance, as we have mentioned, in a nonlinear driven drift wave system [9] a gap solitary wave also coexists with the steady wave, and the summation of them is also a solution of the system. Furthermore, the gap solitary wave may cause a crisis that induces a transition to spatiotemporal chaos [10]. It would be expected that the induced solitons may also play important roles in causing the complex dynamical phenomena of a conservative system.

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